UNSTEADY FLOWS

Table of Contents

1. Unsteady Flow Equations (by Energy Approach)

1.1 Momentum Equation

Herein the derivation of the unsteady flow in Chow (1959) is given. For simplicity, the unsteady flow is treated like steady flow except that an additional variable for the time is used. This time variable takes into account the variation in flow velocity and brings to the acceleration, which produces force and causes additional energy loss.

Consider a change in water surface elevation in the open-channel flow during the time interval Δt as seen in the figure below. The total head *H* at each cross section *i* is defined by

$$H_i = \left(a\frac{V^2}{2g} + y + z \right)_i \tag{1}$$

where *a* = energy correction factor, *V* = velocity averaged over the cross sectional area, *y* = flow depth, and *z* = bottom elevation from a certain datum. Then, the energy balance between two cross sections separated by *dx*, (1) and (2), can be written as

$$H_1 = H_2 + S_H dx + W \tag{2}$$

where S_H = total head gradient and *W* = work done by arc. In order to account for the work done by arc, consider force *F* (per unit volume) such as

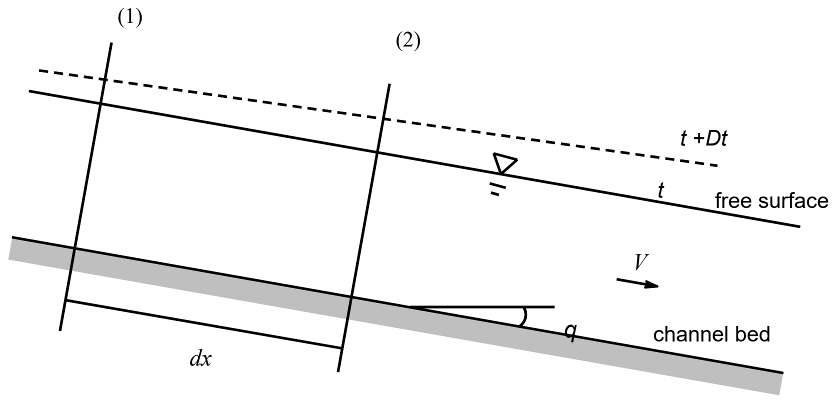


Figure 1. Unsteady flow in a prismatic open channel

$$F = ma = \rho \frac{\partial V}{\partial t} = \frac{\gamma}{g} \frac{\partial V}{\partial t} \tag{3}$$

which makes the force per unit weight to be $\frac{1}{g}\frac{\partial V}{\partial t}$. Therefore, the work done is

$$W = F \cdot dx = \frac{1}{g} \frac{\partial V}{\partial t} dx \tag{4}$$

The total head at section (2) can be expanded by

$$H_2 = a\frac{V^2}{2g} + \frac{\partial}{\partial x}\left(a\frac{V^2}{2g} \right) dx + y + \frac{\partial y}{\partial x} dx + z + dz \tag{5}$$

Substitution of Eqs.(4) and (5) into Eq.(2) and further simplification result in the following momentum equation such as

$$\frac{1}{g}\frac{\partial V}{\partial t} + a\frac{V}{g}\frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} = S_0 - S_H \tag{6}$$

where $S_0 = -\partial z / \partial x$

(Q) Compare the derivation using the Reynolds transport theorem in Chow, Maidment, and Mays (1988).

1.2 Continuity Equation

The change in the amount of water during Δt is approximately

$$T\left(\frac{\partial y}{\partial t}\right)dx\Delta t \tag{7}$$

and the spatial change due to the difference in discharge between two cross sections is

$$\left(\frac{\partial Q}{\partial x}\right)dx\Delta t \tag{8}$$

So the conservation of mass leads to

$$T\left(\frac{\partial y}{\partial t}\right)dx\Delta t + \left(\frac{\partial Q}{\partial x}\right)dx\Delta t = 0 \tag{9}$$

Since $\partial A / \partial t = \partial A / \partial y \cdot \partial y / \partial t = T \cdot \partial y / \partial t$, we have

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{10}$$

which is valid for any arbitrary-shaped cross section.

(Q) Why the set of momentum and continuity equations are called as dynamic equations?

Force is involved in the derivation of the momentum equation.

1.3 Summary of Dynamic Equations

Such a differential equation of the form as

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x}F(k)=0$$

is said to be in conservation form. A differential equation that can be written in conservation form is a conservation law, which states that the time rate of change of the total amount of a substance contained in some region is equal to the inward flux of that substance across the boundaries of that region. The dynamic equations derived in previous sections can be given in either conservative forms or non-conservative forms. This is summarized in the TABLE.

TABLE 9.2.1
Summary of the Saint-Venant equations*

Continuity equation

Conservation form $$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Nonconservation form $$V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} + \frac{\partial y}{\partial t} = 0$$

Momentum equation

Conservation form

$$\frac{1}{A}\frac{\partial Q}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\frac{\partial y}{\partial x} - g(S_o - S_f) = 0$$

Local acceleration term	Convective acceleration term	Pressure force term	Gravity force term	Friction force term

Nonconservation form (unit width element)

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - g(S_o - S_f) = 0$$

Kinematic wave	$g(S_o - S_f)$
Diffusion wave	$g\frac{\partial y}{\partial x} - g(S_o - S_f)$
Dynamic wave	$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - g(S_o - S_f)$

* Neglecting lateral inflow, wind shear, and eddy losses, and assuming $\beta = 1$.

2. Simplification of Dynamic Equations

There are number of ways in approximating the dynamic equation depending upon the importance of each term. That is,

$$\frac{1}{gA}\frac{\partial Q}{\partial t} + \frac{1}{gA}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + \cos\theta\frac{\partial y}{\partial x} - \left(S_0 - S_f\right) = 0 \qquad (11)$$

(1) kinematic wave

(2) non-inertia

(3) quasi-steady dynamic wave

(4) full dynamic wave

An alternative form of the above equation using the primitive variable is

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x}\cos\theta - g(S_0 - S_f) = 0 \tag{12}$$

(1) Kinematic Wave Model

The greatest disadvantage of the kinematic wave approximation is that no backwater effect is considered. Because only one boundary condition at the upstream is necessary. Therefore, the sewer network cannot be analyzed by the kinematic wave approximation.

(2) Non-inertia Model

This is perhaps the most useful among the approximations because it offers a balance between accuracy and simplicity to a large number of field situations. Note that the diffusion coefficient in the non-inertia equation is not a constant but a function $\partial h / \partial x$. While the diffusion coefficient in the kinematic wave equation is a constant. Since the dam break problem can be characterized by strong local acceleration term, non-inertia model does not work. The same is true for hydraulic jump or drop.

(3) Quasi-steady Model

This model requires two boundary conditions, so it does not provide any convenience in the numerical modeling. Also, in general, local and convective acceleration terms have same order of magnitude with opposite sign in the prismatic open channel. Therefore, it is always better (giving accurate results) to ignore both terms instead of neglecting only one term.

(4) Full Dynamic Wave Model

For the free surface flow with Froude number greater than unity (about 1.3), the dynamic equation is not working well because of non-hydrostatic pressure distribution.

For inland water (lake and reservoir), the convective acceleration is small. But in estuary, both local and convective acceleration terms are important.

3. Kinematic Wave Approximation

Kinematic wave vs. Dynamic wave

The motion of an object can be described without considering mass and force, which should be taken account for in the dynamics.

3.1 Kinematic Wave Equation

The motion of wave is described principally by the continuity equation in the kinematic wave theory, where the accelerations and the pressure term are neglected. So the momentum equation becomes

$$S_0 = S_f \tag{13}$$

With the help of Manning's equation, the discharge can be given by

$$Q = \frac{C_m \sqrt{S_0}}{nP^{2/3}} A^{5/3} \tag{14}$$

So the cross-sectional area is

$$A = aQ^{\beta} \tag{15}$$

where

$$a = \left(\frac{nP^{2/3}}{C_m \sqrt{S_0}} \right)^{3/5}$$

$$\beta = 3/5$$

Therefore, we have

$$\frac{\partial Q}{\partial x} + a\beta Q^{\beta-1} \frac{\partial Q}{\partial t} = 0 \tag{16}$$

If the lateral inflow q has to be considered, then Eq.(16) is rewritten as

$$\frac{\partial Q}{\partial x} + a\beta Q^{\beta-1} \frac{\partial Q}{\partial t} = q \tag{17}$$

where q has a dimension of flow rate per length of channel. It is noted that the number of variable is reduced to one owing to the equation of momentum. Once the discharge is obtained by solving the first-order hyperbolic partial differential equation, Eq.(16), the stage or the cross sectional area is estimated by using Eq.(15).

Henderson (1966) showed that Q is a better choice as the dependent variable rather than A. From Eq.(15),

$$\ln A = \ln a + \beta \ln Q$$

or

$$\frac{dQ}{Q} = \frac{1}{\beta}\left(\frac{dA}{A}\right)$$

Using either Manning's or Weisbach's formula, *b* is less than unity, which amplifies the error when *Q* is estimated from *A*.

From Eq.(16), kinematic waves are seen to be resulted from both spatial and temporal changes in *Q*. The total differential of *Q* can be written as

$$dQ = \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial t} dt \tag{18}$$

Then

$$\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t}\frac{dt}{dx} = \frac{dQ}{dx} \tag{19}$$

Comparing Eq.(19) with Eq.(17) leads to that both equations are identical if

$$\frac{dQ}{dx} = q \tag{20}$$

$$\frac{dx}{dt} = \frac{1}{\alpha\beta Q^{\beta-1}} \tag{21}$$

Intuitively, Eq.(20) is true from the definition of the lateral discharge *q*. From Eq.(15),

$$\frac{dQ}{dA} = \frac{1}{\alpha\beta Q^{\beta-1}} \tag{22}$$

Therefore, we have

$$\frac{dx}{dt} = \frac{dQ}{dA} (\equiv c_k) \tag{23}$$

where c_k is the kinematic wave celerity. For a rectangular channel, the kinematic wave celerity is

$$c_k = \frac{1}{B}\frac{dQ}{dy} \tag{24}$$

Lighthill and Whitham (1955) proved that the velocity of the main part of a natural flood wave approximates that of a kinematic wave. There may be several criteria for determining when the kinematic wave approximation is applicable, however, no universal or single criterion exists.

Eq.(23) denotes the characteristics of Eq.(16), the first-order hyperbolic PDE. The equation has only one set of characteristics, along which the disturbance propagates in the downstream direction. The value of Q remains constant along the characteristics without being damped. Eq.(23) is also known as Kleitz –Seddon law and agrees well with observed speeds of flood waves in rivers.

When Manning formula is used for Q in Eq.(22), $\beta = 0.6$ is obtained. Thus a higher value of discharge means higher celerity. This explains how shock generates in the kinematic wave model. If the kinematic wave celerity is constant, then the equation becomes that of linear convection equation describing pure translation of waves without any deformation of wave forms. Therefore, the significant role of the wave celerity which is a nonlinear function of discharge should be addressed in the kinematic wave model.

(Q) Find the characteristic equations of the partial differential equation, Eq.(17).

3.2 Solution of Kinematic Wave

3.2.1 Analytical Solution of Kinematic Wave

Kinematic wave can be analyzed by solving either hyperbolic PDE, Eq.(17), or its characteristic equations, Eqs.(20) and (23). Eq.(17) should be solved numerically because it is a nonlinear PDE in Q. However, the characteristic equations can be solved analytically when there is no lateral flow, i.e., $q = 0$.

If the lateral flow is neglected, then from Eq.(20)

$$\frac{dQ}{dx} = 0 \qquad (25)$$

which means that any particular discharge is conserved along the channel reach. That is, the kinematic wave is a wave of translation without attenuation. In other words, if the flow rate is known at a point in time and space, this flow value is propagating along the channel at the kinematic wave celerity. From Eq.(23),

$$\int_0^x dx = \int_{t_0}^t c_k dt \tag{26}$$

or

$$x = c_k (t - t_0) \tag{27}$$

because c_k is not a function of time. Therefore, the time at which a discharge Q entering a channel of length L at time t_o will appear at the outlet is

$$t = t_0 + L / c_k \tag{28}$$

This solution procedure is possible because the kinematic wave celerity is constant for a given Q, which is true when $q = 0$. Otherwise, Q and the celerity vary along the characteristic lines, which then become curved.

3.2.2 Numerical Solution of Kinematic Wave

(1) Linear Scheme

If the backward difference method is used, then the finite difference form of the space and time derivatives of Q_{i+1}^{j+1} are, respectively,

$$\frac{\partial Q}{\partial x} \approx \frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x} \tag{29}$$

$$\frac{\partial Q}{\partial t} \approx \frac{Q_{i+1}^{j+1} - Q_{i+1}^{j}}{\Delta t} \tag{30}$$

If Q_{i+1}^{j+1} is used to evaluate Q in Eq.(17), then the resulting finite difference equation becomes nonlinear in Q_{i+1}^{j+1}. So the following values for Q and q may be used to make the difference equation linear:

$$Q \approx \frac{1}{2}\left(Q_{i+1}^{j} + Q_{i}^{j+1}\right) \tag{31}$$

$$q \approx \frac{1}{2}\left(q_{i+1}^{j+1} + q_{i+1}^{j}\right) \tag{32}$$

(2) Nonlinear Kinematic Wave Scheme

The finite difference form of eq.(17) can be expressed by

$$\frac{Q_{i+1}^{j+1} - Q_{i}^{j+1}}{\Delta x} + \frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t} = \frac{q_{i+1}^{j+1} - q_{i+1}^{j}}{2} \tag{33}$$

where

$$A_{i+1}^{j+1} = \alpha\left(Q_{i+1}^{j+1}\right)^{\beta} \tag{34}$$

$$A_{i+1}^{j} = \alpha\left(Q_{i+1}^{j}\right)^{\beta} \tag{35}$$

Handouts

Choi, S.-U., and Garcia, M.H. (1993). "Kinematic wave approximation for debris flow routing." *Proceedings of XXV Congress of International Association of Hydraulic Research*, Tokyo, Japan.

Li, R.-M., Simons, D.B., and Stevens, M.A. (1975). "Nonlinear Kinematic Wave Approximation for Water Routing." *Water Resources Research*, 11 (2), 245-252.

Lighthill, M.J., and Whitham, G.B. (1955). "On kinematic waves, I. Flood movement in a long rivers." *Proceedings of Royal Society of London A*, 229 (1178), 281-316.

Ponce, V.M. (1991). "The kinematic wave controversy." *Journal of Hydraulic Engineering*, ASCE, 117(4), 511-525.

4. Non-inertia Approximation

The kinematic wave equation can also be shown to be a form of diffusion equation. So confusion has arisen between two approximations. For the sake of clarity, the approximation by dropping both inertia terms is called by "non-inertia" approximation.

If both local and convective acceleration terms can be ignored, then the momentum equation becomes

$$S_0 - S_f = \cos\theta \frac{\partial y}{\partial x} \tag{36}$$

The discharge can be given by Manning's formula. That is,

$$Q = \frac{C_m A^{5/3}}{n P^{2/3}} \sqrt{S_f}$$

$$= Q_n \left(1 - \frac{\cos\theta}{S_0} \frac{\partial y}{\partial x}\right)^{1/2} \tag{37}$$

where Q_n is the discharge at the normal state defined by

$$Q_n = \frac{C_m \sqrt{S_0}}{n P^{2/3}} A^{5/3} \tag{38}$$

After applying Eq.(37) to Eq.(10), the continuity equation for a rectangular channel is

$$B \frac{\partial y}{\partial t} + \frac{\partial Q_n}{\partial x} - \frac{Q_n \cos\theta}{2 S_0} \frac{\partial^2 y}{\partial x^2} = 0 \tag{39}$$

in which

$$\frac{\partial Q_n}{\partial x} = Bc_k \frac{\partial y}{\partial x} \tag{40}$$

Thus, we have the following diffusion equation:

$$\frac{\partial y}{\partial t} + c_k \frac{\partial y}{\partial x} - \cos\theta \frac{Q_n}{2BS_0} \frac{\partial^2 y}{\partial x^2} = 0 \tag{41}$$

The above equation is a convection diffusion equation with the convection velocity c_k which is equal to the kinematic wave celerity and the diffusion coefficient such as

$$D = \cos\theta \frac{Q_n}{2BS_0} \tag{42}$$

which is a non-linear function of flow depth *y*.

5. Solution of Full-Dynamic Equations

5.1 Method of Characteristics

Let the continuity equation and the momentum equation be denoted by H_1 and H_2, respectively, such as

$$H_1 = \frac{\partial y}{\partial t} + y\frac{\partial V}{\partial x} + V\frac{\partial y}{\partial x} = 0 \tag{43}$$

$$H_2 = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - g\left(S_0 - S_f\right) = 0 \tag{44}$$

Using unknown multiplier λ, H_1 and H_2 can be combined as

$$H = \lambda H_1 + H_2 \tag{45}$$

or

$$H = \left[\frac{\partial V}{\partial x}(V + \lambda y) + \frac{\partial V}{\partial t} \right] + \lambda \left[\frac{\partial y}{\partial x}(V + \frac{g}{\lambda}) + \frac{\partial y}{\partial t} \right] - g(S_0 - S_f) \tag{46}$$

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} + \frac{\partial V}{\partial t} \qquad \text{if} \quad \frac{dx}{dt} = V + \lambda y \tag{47}$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial t} \qquad \text{if} \quad \frac{dx}{dt} = V + \frac{g}{\lambda} \tag{48}$$

Therefore, we have

$$H = \frac{dV}{dt} + \lambda \frac{dy}{dt} - g(S_0 - S_f) = 0 \tag{49}$$

From Eqs.(47) and (48), the unknown multiplier can be obtained as

$$\lambda = \pm\sqrt{g / y} \tag{50}$$

For the positive value of λ,

$$dV + \sqrt{g / y}dy - g(S_0 - S_f)dt = 0 \tag{51}$$

$$dx = (V + \sqrt{gy})dt \tag{52}$$

and for the negative value of λ,

$$dV - \sqrt{g / y}dy - g(S_0 - S_f)dt = 0 \tag{53}$$

$$dx = (V - \sqrt{gy})dt \tag{54}$$

So far we have transformed the hyperbolic system of partial differential equations into a pair of ordinary differential equations, Eqs.(51)-(54).

Numerical analysis of the dynamic equations based upon the method of characteristics is known to yield the most accurate result. It is because of the fact that the method of characteristics involves the most important properties of the hyperbolic partial differential equations. So any numerical technique based upon the method of characteristics seeks numerical solution along the characteristic lines, along which information transmits. In general, finite difference method is used in implementing the method of characteristics. However, a weakness of this method lies in the extension of the numerical scheme into the multi-dimensional problem.

<u>Handouts</u>

Unsteady Flow in Open Channels by Mahmood and Yevjevich

Dynamic Wave Celerity

The wave celerity is the velocity with which the variation in flow travels along the channel. For the dynamic equations, the characteristic equations are

$$\frac{dx}{dt} = V \pm c_d \tag{55}$$

and

$$\frac{d}{dt}(V \pm 2c_d) = g(S_0 - S_f) \tag{56}$$

where c_d is the dynamic wave celerity defined by

$$c_d = \sqrt{gy} \tag{57}$$

for a channel of rectangular cross section. For a channel of arbitrary cross section, c_d is given by

$$c_d = \sqrt{gA/B} \tag{58}$$

The celerity given by Eq.(57) or (58) measures the velocity of the dynamic wave with respect to still water. There are two dynamic waves: one is moving upstream and the other is moving downstream. In order for the wave to propagate up to the channel, c_d should be greater than V, i.e., subcritical flow condition.

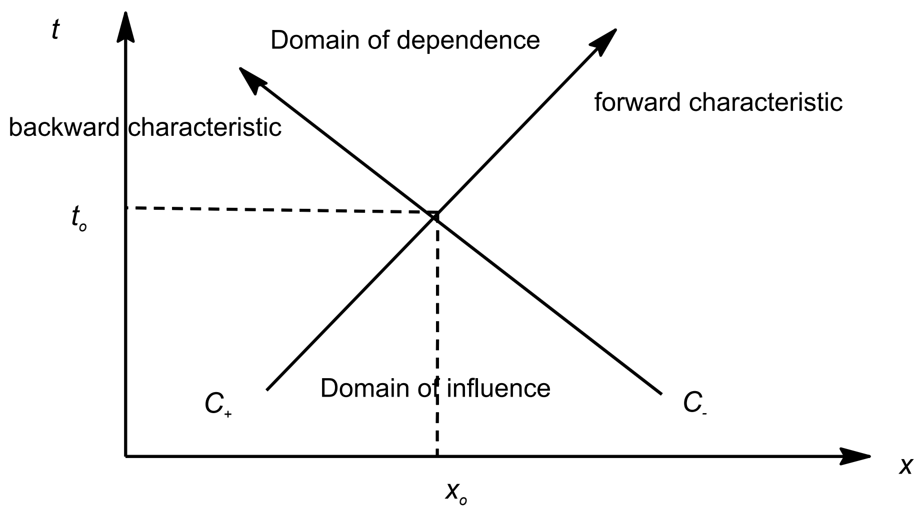


Figure 2. Domain of Dependence and Domain of Influence

Domain of Influence and Domain of Dependence

In the subcritical flow, a disturbance introduced at some point propagates both in upstream and downstream directions. The region included between the two characteristics can possibly experience the influence of the disturbance. This region is called the domain of dependence.

Points outside of this region cannot be influenced by the disturbance.

Conversely, a point $(x_{0,}t_0)$ can be affected by disturbances from earlier times originating from the domain of influence. The propagation of disturbances from points outside of this region is not fast enough to reach x_o before or at time t_o. Of course they will reach this point at some later time.

(Q) Explain the properties of characteristics for the supercritical flow?
Each point depends only on upstream disturbances and influences only downstream points.

5.2 Implicit Dynamic Wave Model

The method is perfectly described in Chow, Maidment , and Mays (1988). Many references are included therein. However, the first step is made by the authors of the following papers:

Amein, M., and Fang, C.S. (1970). ‘‘Implicit flood routing in natural channels.’’ *Journal of The Hydraulics Division*, ASCE, 96(HY12), 2481-2500.

Amein, M., and Chu, H.-L. (1975). ‘‘Implicit numerical modeling of unsteady flows.’’ *Journal of The Hydraulics Division*, ASCE, 101(HY6), 717-731.

TABLE II

EXPLICIT FINITE-DIFFERENCE SCHEMES

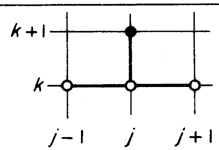
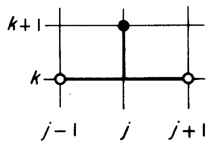
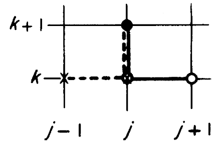
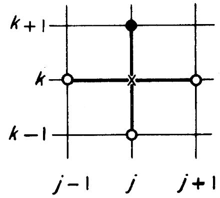
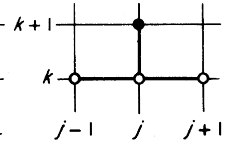
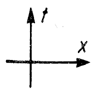
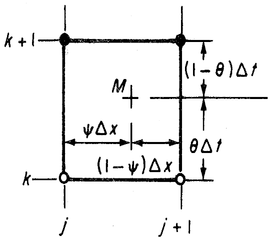
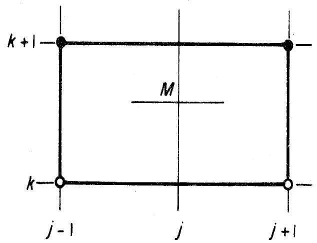
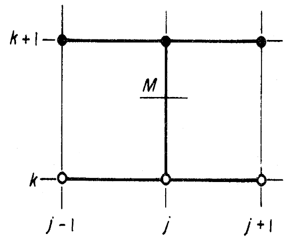
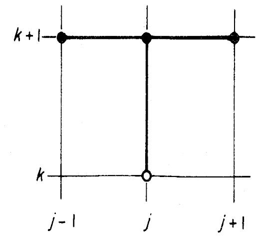
Computational grid-point structure	Unstable	Diffusive	L-shaped (upstream)	Leap-frog	Lax-Wendroff
structure (●) Unknown (o, x) Known (t ↑, x →)					
$\Delta x = x_{j+1} - x_j$ $= x_j - x_{j-1}$ $\Delta t = t_{k+1} - t_k$ $= t_k - t_{k-1}$	$\frac{\partial W}{\partial x} \approx \frac{W_{j+1}^k - W_{j-1}^k}{2\,\Delta x}$	$\frac{W_{j+1}^k - W_{j-1}^k}{2\,\Delta x}$	$\frac{W_{j+1}^k - W_j^k}{\Delta x}$ or $\frac{W_j^k - W_{j-1}^k}{\Delta x}$	$\frac{W_{j+1}^k - W_{j-1}^k}{2\,\Delta x}$	Depends on PDE s used; commonly includes $\frac{W_{j+1}^k - W_{j-1}^k}{2\,\Delta}$
	$\frac{\partial^2 W}{\partial x^2}$				$\frac{W_{j+1}^k - 2W_j^k + W_{j-1}^k}{(\Delta x)^2}$
Discretization expressions	$\frac{\partial W}{\partial t} \approx \frac{W_j^{k+1} - W_j^k}{\Delta t}$	$\frac{W_j^{k+1} - \frac{W_{j+1}^k + W_{j-1}^k}{2}}{\Delta t}$	$\frac{W_j^{k+1} - W_j^k}{\Delta t}$	$\frac{W_j^{k+1} - W_j^{k-1}}{2\,\Delta t}$	
	$W \approx W_j^k$	W_j^k or $\frac{W_{j+1}^k + W_{j-1}^k}{2}$	W_j^k	W_j^k or $\frac{W_{j+1}^k + W_{j-1}^k}{2}$	W_j^k
Discretization error	$O[\Delta^2]$	$O[\Delta^2]$	$O[\Delta^2]$	$O[\Delta^3]$	$O[\Delta^3]$

TABLE III

IMPLICIT FINITE-DIFFERENCE SCHEMES[a]

Computational grid-point structure	Box	Rectangle	Wide flange	Tee
(●) Unknown (○) Known  $\Delta x = x_{j+1} - x_j$ $\Delta t = t_{k+1} - t_k$				
Discretization expressions	$\frac{\partial W}{\partial x} \approx \theta \frac{W_{j+1}^{k+1} - W_j^{k+1}}{\Delta x} + (1-\theta)\frac{W_{j+1}^k - W_j^k}{\Delta x}$	$\theta \frac{W_{j+1}^{k+1} - W_{j-1}^{k+1}}{x_{j+1} - x_{j-1}} + (1-\theta)\frac{W_{j+1}^k - W_{j-1}^k}{x_{j+1} - x_{j-1}}$	The same as rectangle schemes	$\frac{W_{j+1}^{k+1} - W_{j-1}^{k+1}}{2\Delta x}$
	$\frac{\partial W}{\partial t} \approx \psi \frac{W_{j+1}^{k+1} - W_{j+1}^k}{\Delta t} + (1-\psi)\frac{W_j^{k+1} - W_j^k}{\Delta t}$	$q \frac{W_{j+1}^{k+1} - W_{j+1}^k}{\Delta t} + (1-q)\frac{W_{j-1}^{k+1} - W_{j-1}^k}{\Delta t}$	$\frac{W_j^{k+1} - W_j^k}{\Delta t}$	$\frac{W_j^{k+1} - W_j^k}{\Delta t}$
	$W \approx \chi\left[\psi W_{j+1}^{k+1} + (1-\psi) W_j^{k+1}\right] + (1-\chi)\left[\psi W_{j+1}^k + (1-\psi) W_j^k\right]$ in which $0.5 \le \theta \le 1.0$; $0 < \psi < 1.0$, $0 < \chi < 1.0$	$\chi\left[q W_{j+1}^{k+1} + (1-q) W_{j-1}^{k+1}\right] + (1-\chi)\left[q W_{j+1}^k + (1-q) W_{j-1}^k\right]$ in which $q = \frac{x_j - x_{j-1}}{x_{j+1} - x_{j-1}}$	The same as rectangle schemes	W_j^k or W_j^{k+1}

[a] Discretization error varies with values and combinations of weighting factors. Aside from stability and other numerical and physical considerations, better accuracy is usually obtained by a more symmetrical arrangement.